Problem Set III: Due by last class. N.B.: It may be advantageous to do the problems out of order.

1.) A spot on a hot surface produces thermal convection above it. The convection is turbulent, and causes a fixed net heat flux Q, upward.

a.) By balancing turbulent dissipation with buoyant production, estimate a typical turbulent velocity V_T. Calculate $\left\langle V_T(z)^2 \right\rangle$ and $\left\langle \tilde{T}(z)^2 \right\rangle$.

b.) Using mixing length arguments, determine the vertical profile of the mean temperature $\langle T(z) \rangle$.

2.) Consider magnetic buoyancy interchange instabilities as discussed in class. Assume entropy stratification is neutral, so $dS_0/dz = 0$. Take η small, but non-zero.

a.) Use quasilinear theory to calculate the vertical flux of magnetic intensity. Since, $\Gamma \sim -\partial_z \ell n(\langle B \rangle / \rho)$, show that Γ may be written as

$$\Gamma = -D \frac{\partial \langle B \rangle}{\partial z} + V \langle B \rangle.$$

Calculate D, V. Interpret your result. For $\rho = \rho_0(z)$. What profile corresponds to the zero flux state?

b.) What is the origin of the pinch velocity V? Explain its significance.

c.) As a related example, consider evolution of the particle density according to

$$\partial n / \partial t + \nabla \cdot (n \underline{V}) = 0.$$

Take $n_0 = n_0(x)$, $\underline{B} = B_0(x)\hat{z}$ and $\underline{V} = -\nabla \phi x \hat{z} / B_0(x)$.

Show that density evolution can be related to the incompressible advection of the field n/B:

$$\partial n/\partial t + \underline{\mathrm{V}}_{eff} \cdot \underline{\nabla}(n/B) = 0$$

where $\underline{\nabla} \cdot \underline{\mathrm{V}}_{eff} = 0$.

Show that the mean field equation for $\langle n \rangle$ obeys:

$$\frac{\partial}{\partial t}\langle n \rangle = \frac{\partial}{\partial x} D \frac{\partial}{\partial x}\left(\frac{\langle n \rangle}{\langle B \rangle}\right)$$

where we took $\langle n/B \rangle \cong \langle n \rangle / \langle B \rangle$. Discuss the zero flux state here. What are its implications for the density profile?

Re-write the mean field equation as

$$\frac{\partial \langle n \rangle}{\partial t} = -\frac{\partial}{\partial x}\left[-D\frac{\partial \langle n \rangle}{\partial x} + \mathrm{V}\langle n \rangle\right].$$

Relate D and V, here. Under what circumstances will V be inward, i.e. *up* the density gradient?

d.) Relate the results of parts b.), c.) here. What is the lesson?

Congratulations! You have just developed the basics of TEP pinch theory!

3.) Consider a fluid in hydrostatic equilibrium with a vertical entropy gradient $\partial S/\partial z < 0$. Take $\underline{g} = -g\hat{z}$.

a.) Starting from the basic equations, derive the growth rate of ideal Rayleigh-Bernard instability. You will find it helpful to relate the density perturbation $\tilde{\rho}/\rho_0$ to the temperature perturbation $\tilde{T}/T_0$ by exploiting the fact that the instability develops slowly in comparison to the sound transit time across a cell. Relate your result to the Schwarzschild criterion discussed in class.

b.) Now, include thermal diffusivity (χ) and viscosity (ν) in your analysis. Calculate the critical temperature gradient for instability, assuming $\chi \sim \nu$. Discuss how this compares to the ideal limit. What happens if $\nu > \chi$?

4.) Now again, consider the system of Problem 2, now immersed in a uniform magnetic field $\underline{B} = B_0\hat{z}$.

a.) Assuming ideal dynamics, use the Energy Principle to analyze the stability of a convection cell of vertical wavelength k_z . Of course, $k_z L_p >> 1$, where L_p is a mean pressure scale length. What is the effect of the magnetic field? Can you estimate how the growth rate changes?

b.) Now, calculate the growth rate using the full MHD equations. You may assume $\underline{\nabla} \cdot \underline{V} = 0$. What structure convection cell is optimal for vertical transport of heat when B_0 is strong? Explain why. What happens when $B_0 \to \infty$? Congratulations - you have just derived a variant of the Taylor-Proudman theorem!

5.) This problem asks you to explore the Current Convective Instability (CCI) in a homogeneous medium and its sheared field relative, the Rippling Instability.

a.) Consider first a current carrying plasma in a straight magnetic field $\underline{B} = B_0\hat{z}$ - i.e. ignore the poloidal field, etc. Noting that the resistivity η is a function of temperature (ala' Spitzer - c.f. Kulsrud 8.7), calculate the electrostatic resistive instability growth rate, assuming T evolves according to:

$$\frac{\partial T}{\partial t} + \underline{v} \cdot \underline{\nabla} T - \chi_{\parallel} \partial_z^2 T - \chi_{\perp} \nabla_{\perp}^2 T = 0$$

and the electrostatic Ohm's Law is just

$$-B_0 \partial_z \phi = \frac{1}{\eta} \frac{d\eta}{dt} \tilde{T} (\eta J_0).$$

b.) *Thoroughly* discuss the physics of this simple instability, i.e.

- what is the free energy source?
- what is the mechanism?
- what are the dampings and how do they restrict the unstable spectrum?
- how does spectral asymmetry enter?
- what is the cell structure?

c.) Use quasilinear theory and the wave breaking limit to estimate the heat flux from the C.C.I.

d.) Now, consider the instability in a *sheared* magnetic field.

i.) What difficulties enter the analysis?

ii.) Resolve the difficulty by considering coupled evolution of vorticity, Ohm's Law (in electrostatic limit but with temperature fluctuations) and electron temperature. Compute the growth rate in the limit $\chi_{\parallel}, \chi_{\perp} \to 0$. Compute the mode width. Discuss how asymmetry enters here. Explain why.

e.) Noting that $\chi_{\parallel} >> \chi_{\perp}$ (why? - see Kulsrud 8.7), estimate when parallel thermal conduction becomes an important damping effect. Can $\chi_{\parallel}$ alone ever absolutely stabilize the rippling mode?

f.) Calculate the quasilinear heat flux and use the breaking limit to estimate its magnitude.

6.) *Taylor in Flatland*

Taylor awakes one morning, and finds himself in Flatland, a 2D world. Seeking to relax, he sets about reformulating his theory for that planar universe.

a.) Write down the visco-resistive 2D MHD equations, and show that *three* quadratic quantities are conserved, as $\eta \to 0$, $\nu \to 0$.

b.) Which of these is the most likely to constrain magnetic relaxation? Argue that

i.) the local version of this quantity is conserved for an 'flux circle', as $\eta \to 0$,

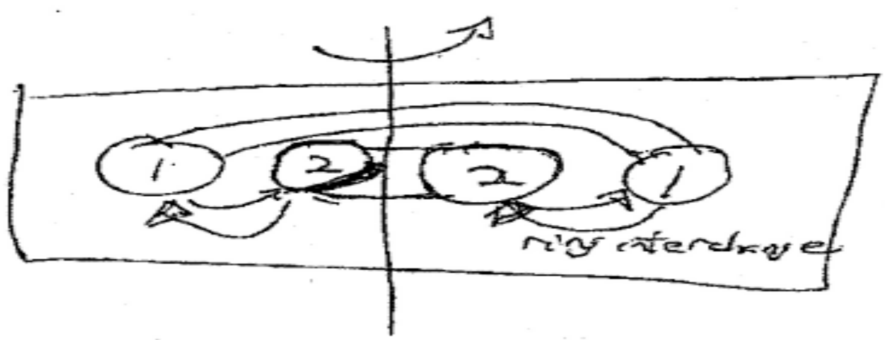
ii.) the global version is the most "rugged", for finite η.

c.) Formulate a 2D Taylor Hypothesis - i.e. that magnetic energy is minimized while the quantity you identified from b.) ii.) is conserved. What equation describes this state? Show that the solution is force-free. What quantity is constant in Flatland? Hence, what is the endstate of Taylor relaxation in 2D?

d.) Consider the possibility that $\nu >> \eta$ in Flatland. Derive the mean field evolution equation for mean magnetic potential. Discuss!

e.) *Optional - Extra Credit* - Describe the visit of the Terrifying Torus to Flatland. How would 2D Taylor perceive this apparition?
N.B. You may find it useful to consult *Flatland*, by E. Abbott.

7.) Consider a rotating fluid with mean $\underline{V} = r\Omega(r)\hat{\theta}$. Your task here is to analyze the stability of this system to interchanges of 'rings', i.e.



In all cases, assume $\underline{\nabla} \cdot \underline{V} = 0$ and $k_\theta = 0$, so the interchange motions carry no angular momentum themselves and the cells sit in the *r*-*z* plane.

a.) At the level of a "back-of-an-envelope" calculation, calculate the change in energy resulting from the incompressible interchange of rings (1) and (2). Note that $E = L^2/2mr^2$ and that the angular momentum L of an interchanged ring is conserved, since $k_\theta = 0$. From this, what can you conclude about the profile of $\Omega(r)$ necessary for stability? Congratulations - you have just derived the Rayleigh criterion!

b.) Now, calculate the interchange growth rate by a direct solution of the fluid equations. You may find it helpful to note that for rotating fluids in cylindrical geometry:

$$\frac{\partial V_r}{\partial t} + \underline{V} \cdot \underline{\nabla} V_r - \frac{V_\theta^2}{r} = \frac{-1}{\rho}\frac{\partial P}{\partial r},$$

$$\frac{\partial V_\theta}{\partial t} + \underline{V} \cdot \underline{\nabla} V_\theta + \frac{V_r V_\theta}{r} = \frac{-1}{\rho r}\frac{\partial P}{\partial \theta},$$

$$\frac{\partial V_z}{\partial t} + \underline{V} \cdot \underline{\nabla} V_z = \frac{-1}{\rho}\frac{\partial P}{\partial z}.$$

Show that you recover the result of part (a).

c.) Compare and contrast this interchange instability to an incompressible Rayleigh-Taylor instability. Make a table showing the detailed correspondences.

8.) Show explicitly the relation between:

a.) the vorticity flux and the $\underline{E} \times \underline{B}$ velocity Reynolds stress.

b.) $\langle \tilde{B}_r \tilde{J}_{\parallel e} \rangle$ and the magnetic Reynolds (Maxwell) stress.